Updating and downdating algorithms for polynomial least squares fitting

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CEA/DIF

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Introduction

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- The inverse eigenvalue problem is to construct J_m from the spectral data $t_j, w_j, j = 1, \ldots, m$
- The eigenvalues of J_m must be the nodes t_j and the first elements of the eigenvectors must be w_j

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- $d^m = P_m D_m^2 y^m$, $\quad d^n = P_{n,m} D_m^2 y^m$, $n < m$

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- This is done by constructing the Jacobi matrix defined by the coefficients of the three-term recurrence

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Updating

This was considered by Elhay, Golub and Kautsky (1991)

Theorem

Let $\sigma_m = (w_1^2 + \cdots + w_m^2)^{1/2}$. *The solution is*

$$J_{m+1} = Q \begin{pmatrix} J_m & 0 \\ 0 & t_{m+1} \end{pmatrix} Q^T$$

$$\sigma_{m+1} = (\sigma_m^2 + w_{m+1}^2)^{1/2}$$

$$d^{m+1} = Q \begin{pmatrix} d^m \\ w_{m+1} y_{m+1} \end{pmatrix}$$

Q orthogonal matrix uniquely determined by requiring J_{m+1} *to be tridiagonal and Q to be such that*

$$Q(\sigma_m e^1 + w_{m+1} e^{m+1}) = \sigma_{m+1} e^1$$

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- R_1 computed to have $Q(\sigma_m e^1 + w_{m+1} e^{m+1}) = \sigma_{m+1} e^1$
- This creates nonzero elements in the last row (and last column) that are chased by subsequent rotations $R_2, \ldots, R_m$ to make J_{m+1} tridiagonal

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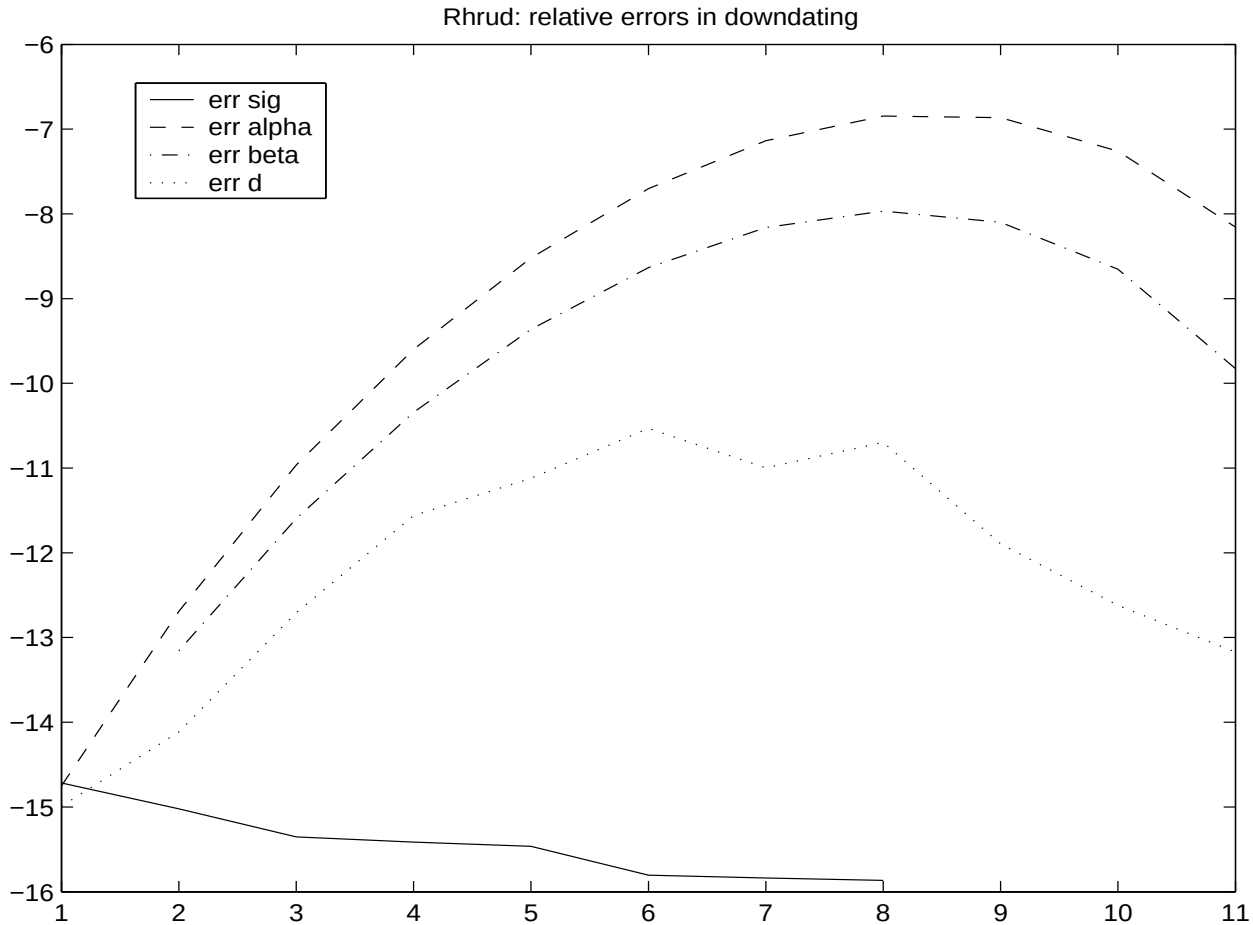
We downdate by removing one node at a time and compare $(\alpha, \beta, \sigma, d)$ to what we obtained when updating

Updating with rot. – Downdating with hyp. rot.



Example 1: $\log_{10}$ of relative errors when downdating, $N = 12$

Reversed rotations

Theorem (Elhay, Golub, Kautsky)

Given σ_{m+1}, J_{m+1} and d^{m+1}, the solution when removing the triplet $\{t_{m+1}, w_{m+1}, y_{m+1}\}$ from the data is

$$\begin{pmatrix} \tilde{J}_m & 0 \\ 0 & t_{m+1} \end{pmatrix} = Q J_{m+1} Q^T$$

$$\tilde{\sigma}_m = (\sigma_m^2 - w_{m+1}^2)^{1/2}, \quad \begin{pmatrix} \tilde{d}^m \\ w_{m+1} y_{m+1} \end{pmatrix} = Q d^{m+1}$$

where the orthogonal matrix Q is uniquely determined by requiring $\tilde{J}_m$ to be tridiagonal and such that

$$Q(\sigma_{m+1} e^1 - w_{m+1} q) = \tilde{\sigma}_m e^1$$

q being the normalized eigenvector of J_{m+1} corresponding to t_{m+1} scaled to have a positive first element.

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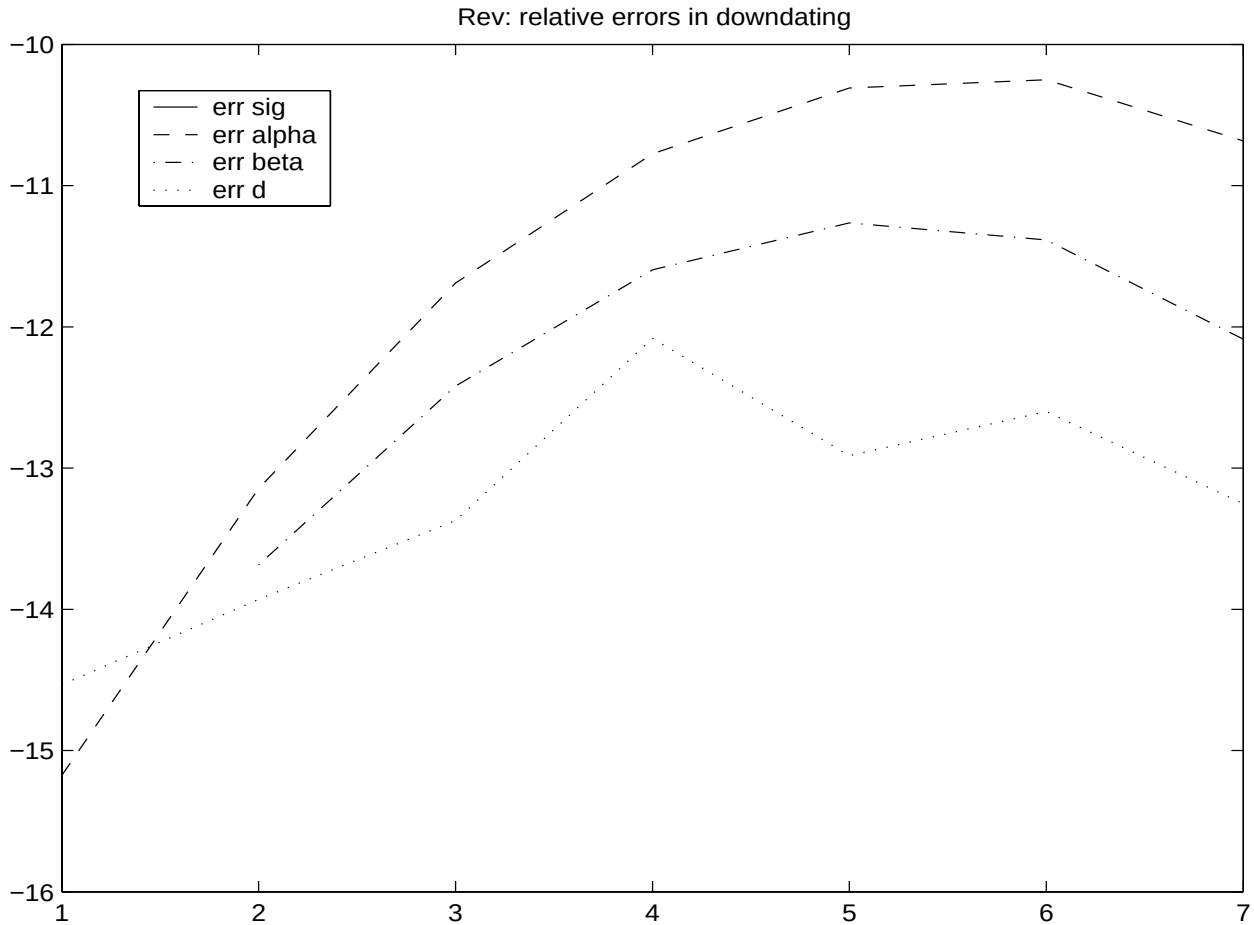
- This does not work well...
- All these computations break down (sqrt of negative numbers) when n is moderately large

Updating with rot. – Downdating with REV.



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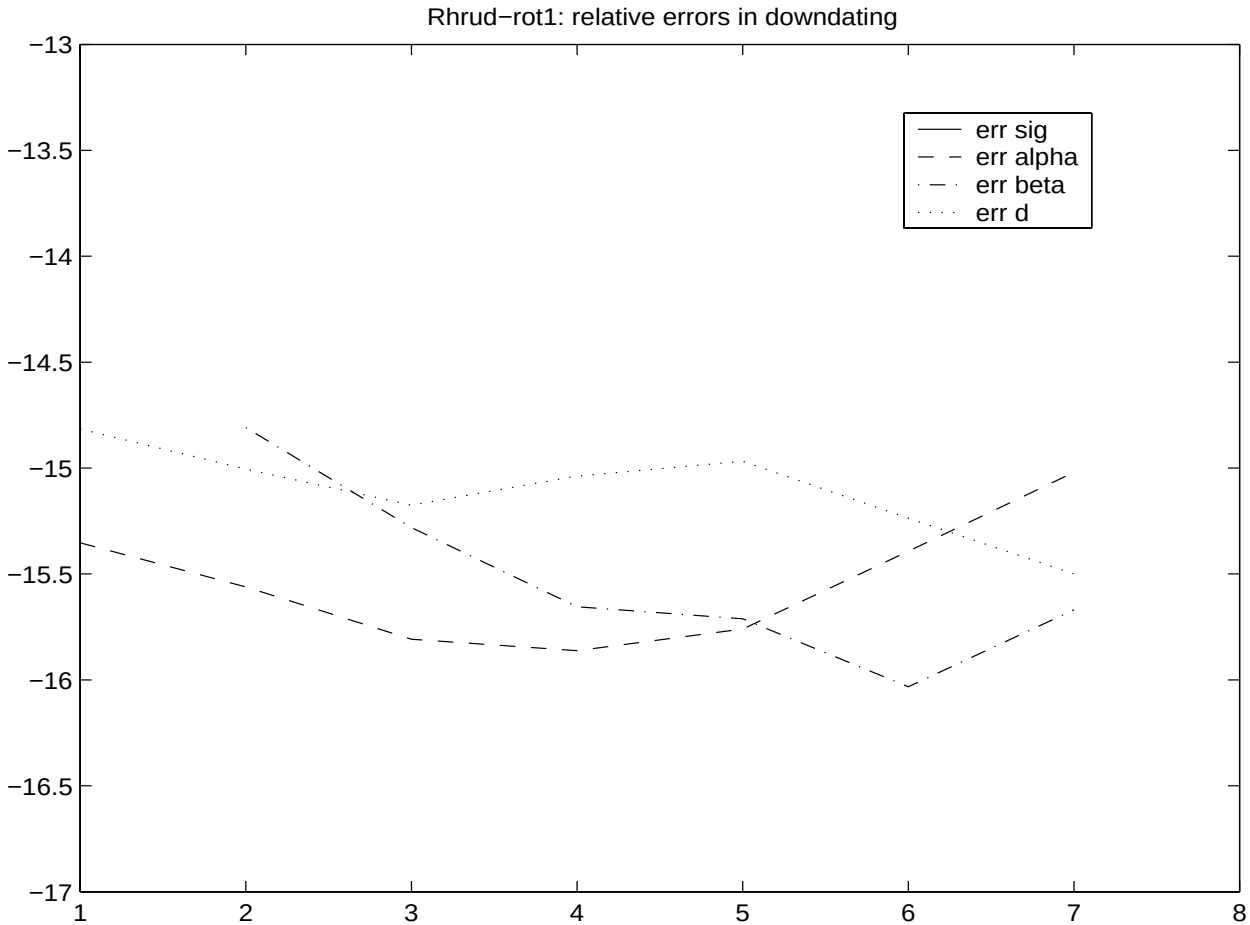
- We can use large data sets without problem
- However, this is too costly

Updating with rot. – Downdating with eigenvectors



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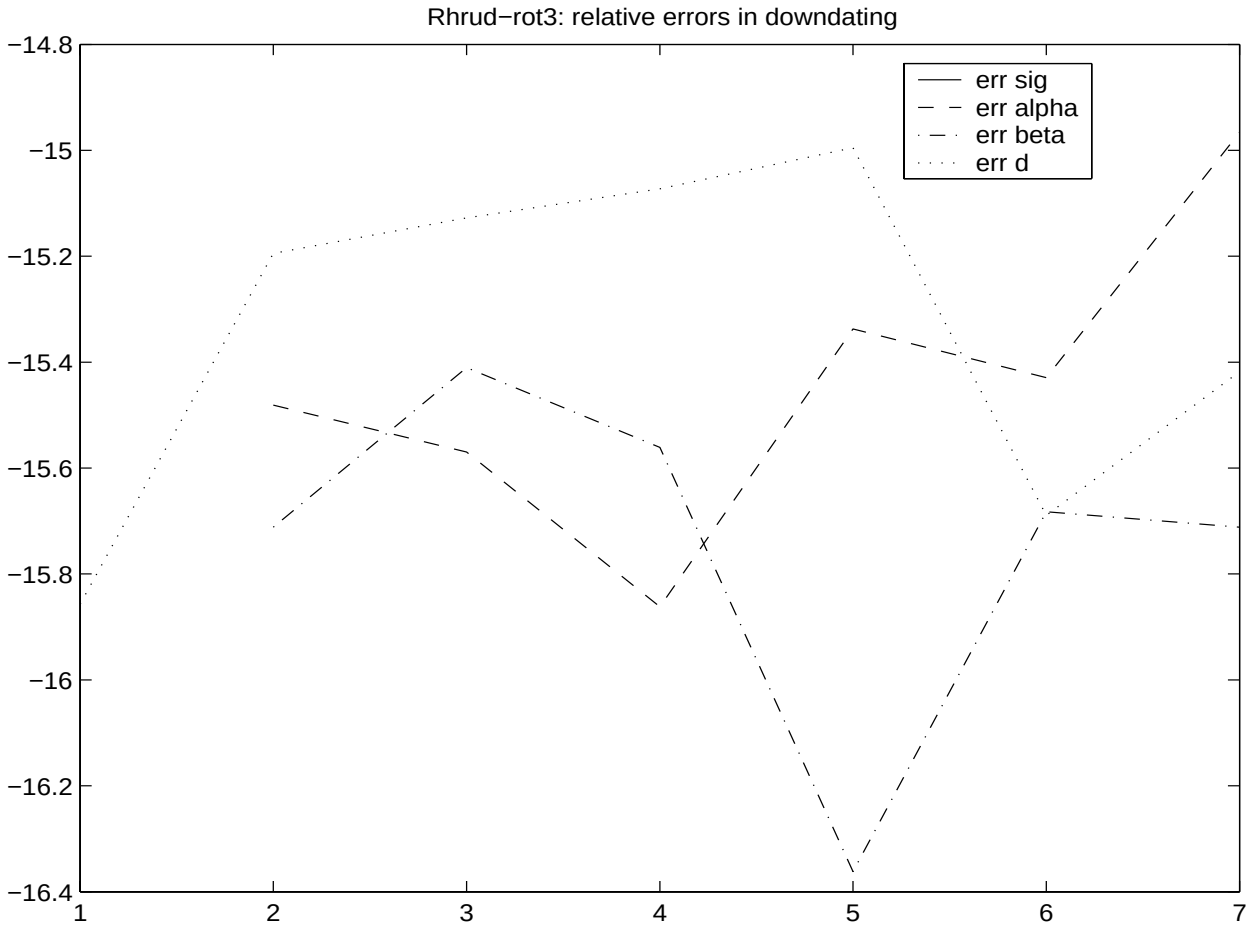
Updating with rot. – Downdating with MR^3



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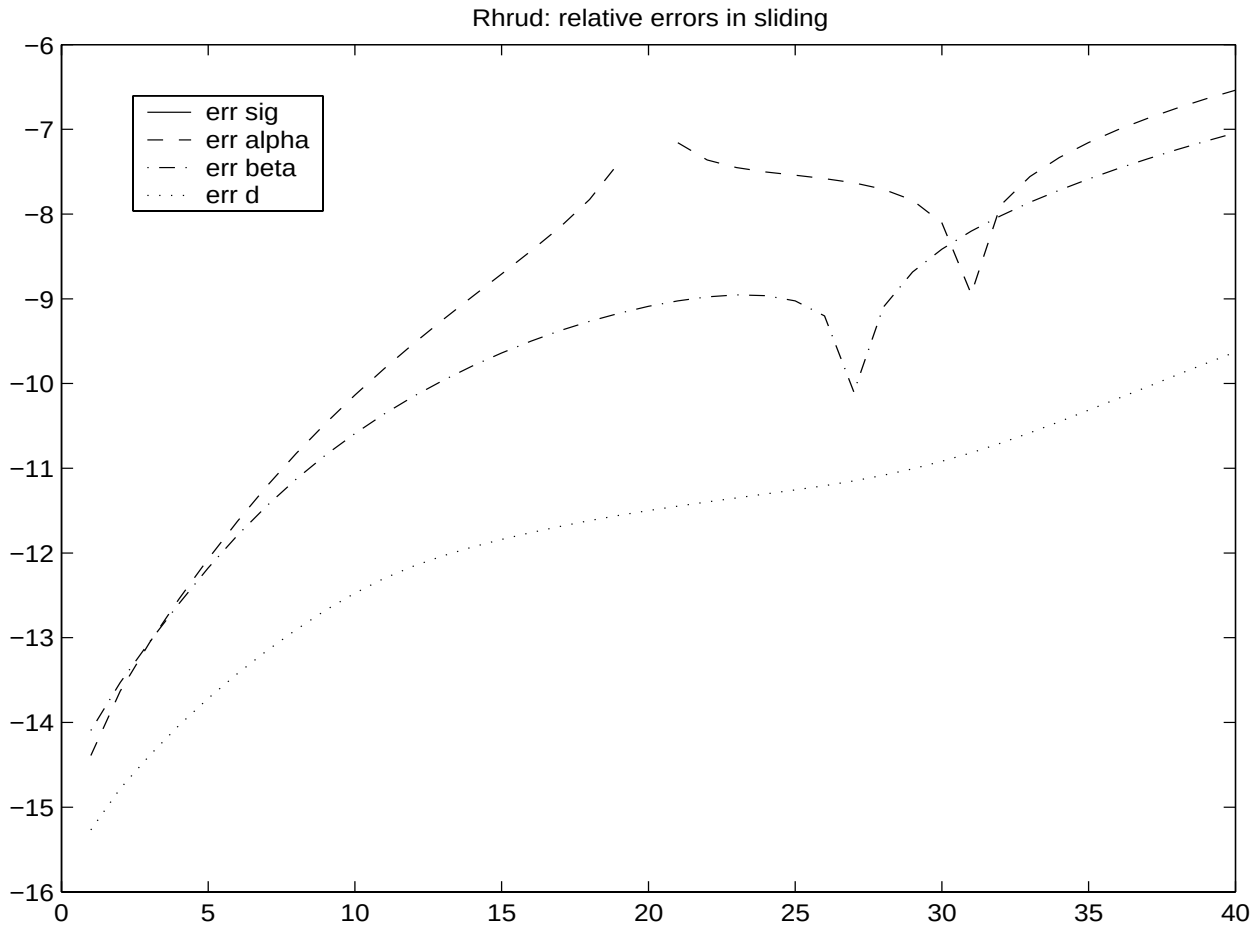
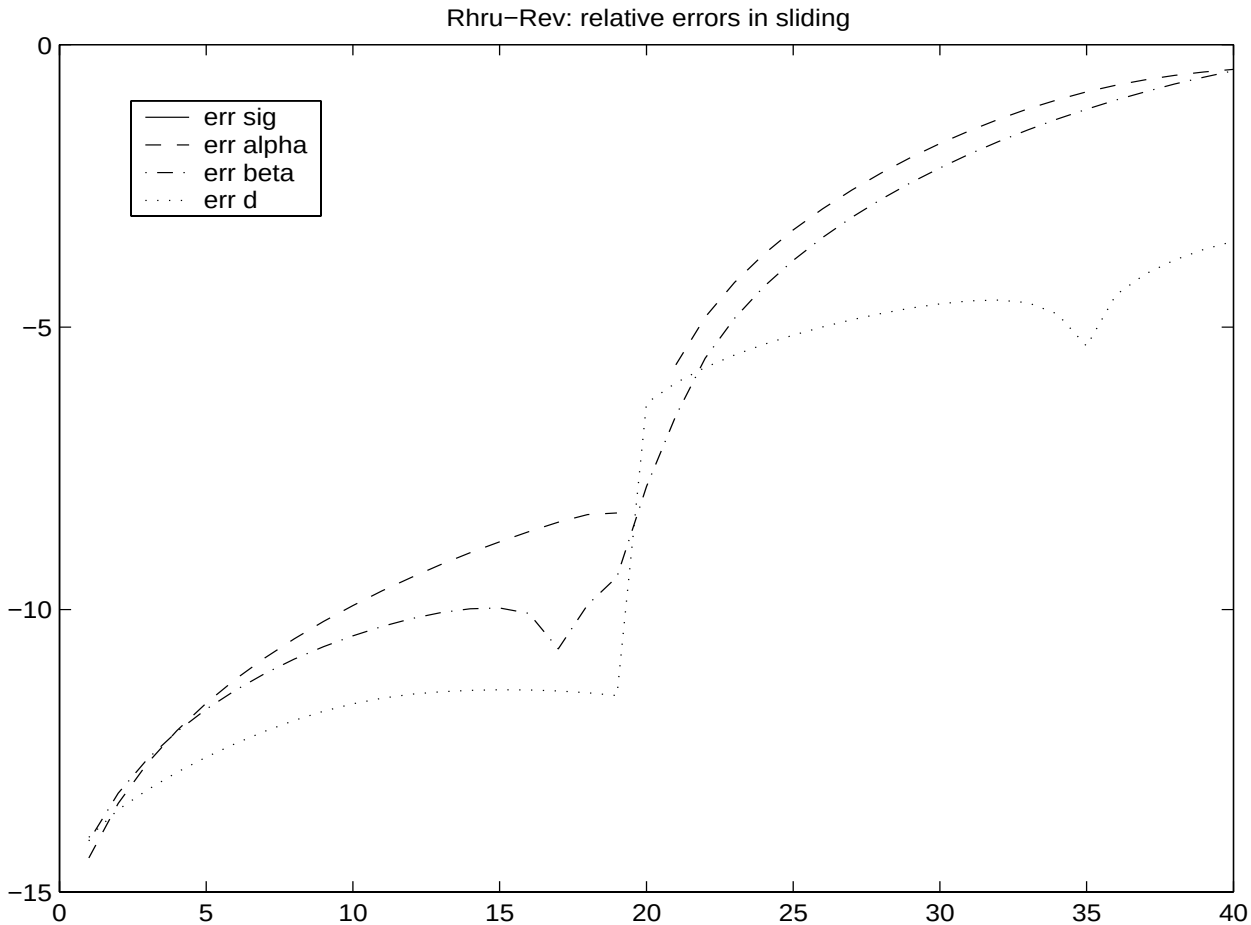
- For $k = 2, \ldots, N - M + 1$ we add $\{t_{k+M}, w_{k+M}, y_{k+M}\}$ and then remove $\{t_{k-1}, w_{k-1}, y_{k-1}\}$
- We compare to the solution computed directly (using RHR) for Y_k

Updating – Downdating with RHRud



Example 1: $\log_{10}$ of relative errors when sliding the window, $N = 50, M = 10, n = 5$

Updating with rot. – Downdating with REV



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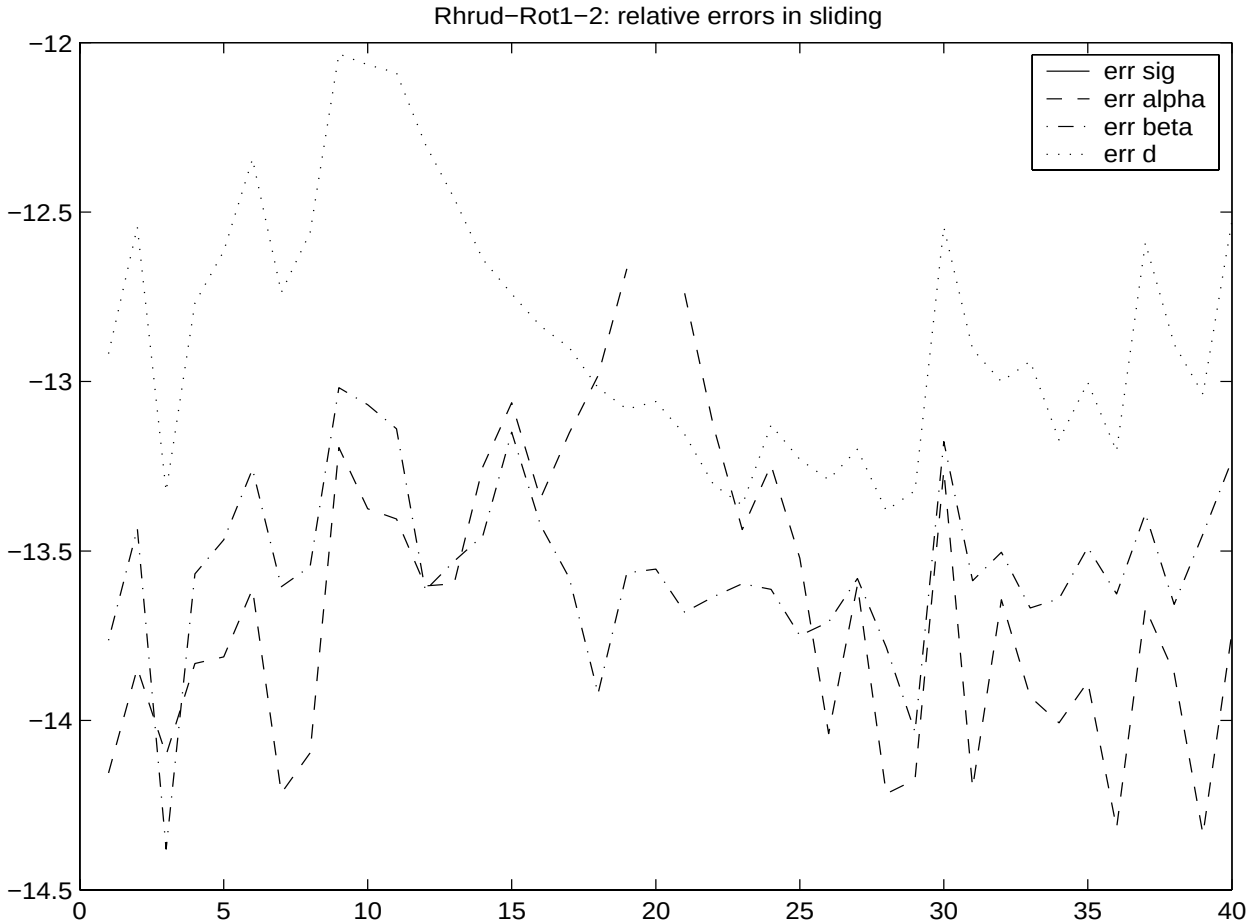

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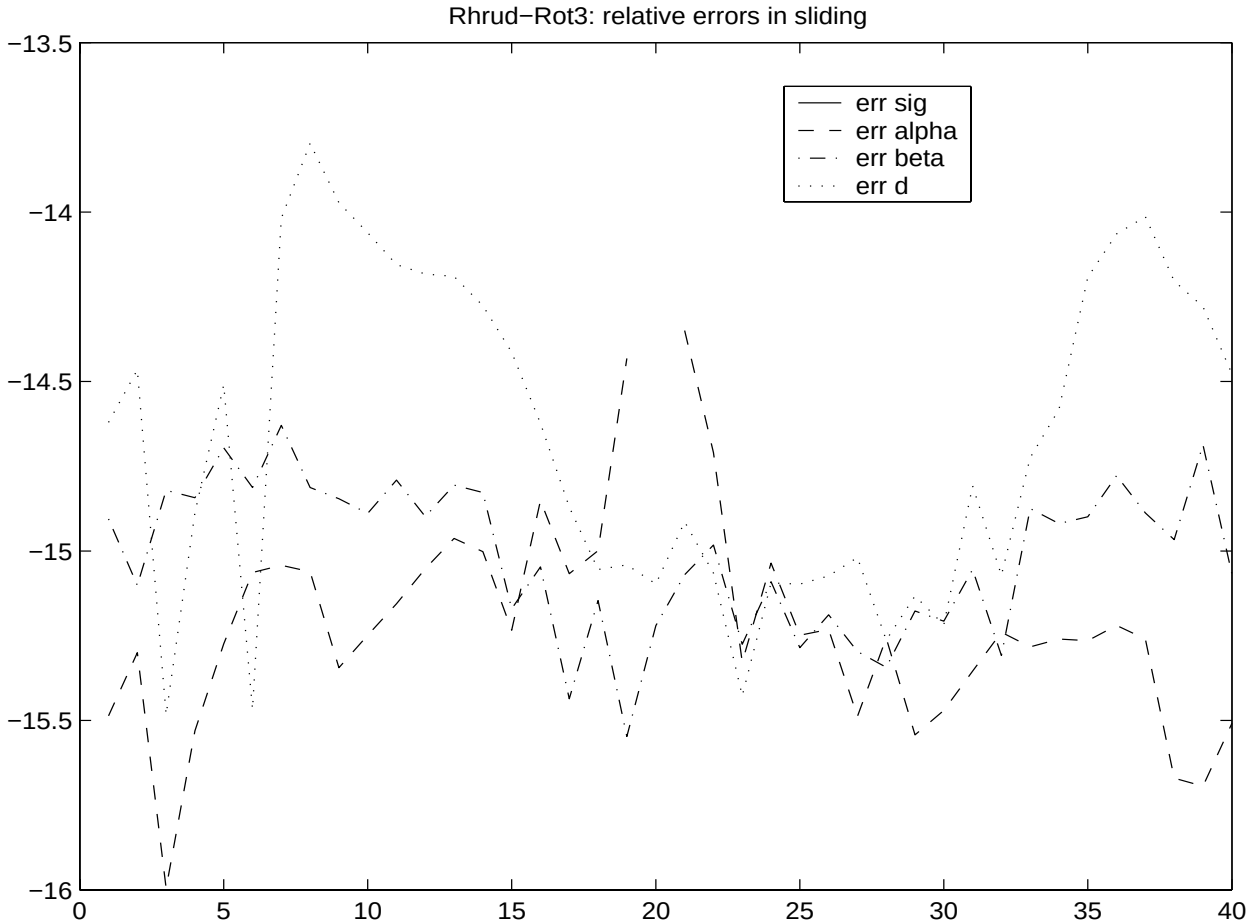

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- Using these modifications, large data sets can be handled